

GROMOV HYPERBOLICITY OF MINOR GRAPHS

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ABSTRACT. If X is a geodesic metric space and $x_1, x_2, x_3 \in X$, a *geodesic triangle* $T = \{x_1, x_2, x_3\}$ is the union of the three geodesics $[x_1x_2]$, $[x_2x_3]$ and $[x_3x_1]$ in X . The space X is δ -*hyperbolic* (in the Gromov sense) if any side of T is contained in a δ -neighborhood of the union of the two other sides, for every geodesic triangle T in X . The study of hyperbolic graphs is an interesting topic since the hyperbolicity of a geodesic metric space is equivalent to the hyperbolicity of a graph related to it. In the context of graphs, to remove and to contract an edge of a graph are natural transformations. The main aim in this work is to obtain quantitative information about the distortion of the hyperbolicity constant of the graph $G \setminus e$ (respectively, G/e) obtained from the graph G by deleting (respectively, contracting) an arbitrary edge e from it. This work provides information about the hyperbolicity constant of minor graphs.

Keywords: Hyperbolic graph; minor graph; edge contraction; deleted edge.

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1. INTRODUCTION

Hyperbolic spaces play an important role in geometric group theory and in the geometry of negatively curved spaces (see [1, 18, 19]). The concept of Gromov hyperbolicity grasps the essence of negatively curved spaces like the classical hyperbolic space, Riemannian manifolds of negative sectional curvature bounded away from 0, and of discrete spaces like trees and the Cayley graphs of many finitely generated groups. It is remarkable that a simple concept leads to such a rich general theory (see [1, 18, 19]).

The first works on Gromov hyperbolic spaces deal with finitely generated groups (see [19]). Initially, Gromov spaces were applied to the study of automatic groups in the science of computation (see, *e.g.*, [28]); indeed, hyperbolic groups are strongly geodesically automatic, *i.e.*, there is an automatic structure on the group [12].

The concept of hyperbolicity appears also in discrete mathematics, algorithms and networking. For example, it has been shown empirically in [36] that the internet topology embeds with better accuracy into a hyperbolic space than into an Euclidean space of comparable dimension; the same holds for many complex networks, see [24]. A few algorithmic problems in hyperbolic spaces and hyperbolic graphs have been considered in recent papers (see [14, 16, 17, 23]). Another important application of these spaces is the study of the spread of viruses through on the internet (see [21, 22]). Furthermore, hyperbolic spaces are useful in secure transmission of information on the network (see [20, 21, 22, 27]). The hyperbolicity has also been applied in the field of random networks. For example, it was shown in [34, 35] that several types of small-world networks and networks with given expected degrees are not hyperbolic in some sense.

The study of Gromov hyperbolic graphs is a subject of increasing interest in graph theory; see, *e.g.*, [2, 3, 4, 5, 6, 7, 9, 11, 15, 20, 21, 22, 24, 26, 27, 29, 30, 31, 32, 37, 39] and the references therein.

We say that the curve γ in a metric space X is a *geodesic* if we have $L(\gamma|_{[t,s]}) = d(\gamma(t), \gamma(s)) = |t - s|$ for every $s, t \in [a, b]$ (then γ is equipped with an arc-length parametrization). The metric space X is said *geodesic* if for every couple of points in X there exists a geodesic joining them; we denote by $[xy]$ any geodesic joining x and y ; this notation is ambiguous, since in general we do not have uniqueness of geodesics, but it is very convenient. Consequently, any geodesic metric space is connected. If the metric space X is a graph, then the edge joining the vertices u and v will be denoted by $[u, v]$.

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In order to consider a graph G as a geodesic metric space, we identify (by an isometry) any edge $[u, v] \in E(G)$ with the interval $[0, 1]$ in the real line; then the edge $[u, v]$ (considered as a graph with just one edge) is isometric to the interval $[0, 1]$. Thus, the points in G are the vertices and, also, the points in the interior of any edge of G . In this way, any connected graph G has a natural distance defined on its points, induced by taking shortest paths in G , and we can see G as a metric graph. Throughout this paper, $G = (V, E)$ denotes a connected graph such that every edge has length 1 and $V \neq \emptyset$. These properties guarantee that any connected graph is a geodesic metric space. We will work both with simple and non-simple graphs. The difference between them is that the first type does not contain either loops or multiple edges. Although the operation of contraction is naturally defined for non-simple graphs, simple graphs are a more usual context in the study of hyperbolicity.

If X is a geodesic metric space and $x_1, x_2, x_3 \in X$, the union of three geodesics $[x_1x_2]$, $[x_2x_3]$ and $[x_3x_1]$ is a *geodesic triangle* that will be denoted by $T = \{x_1, x_2, x_3\}$ and we will say that x_1, x_2 and x_3 are the vertices of T ; it is usual to write also $T = \{[x_1x_2], [x_2x_3], [x_3x_1]\}$. We say that T is δ -thin if any side of T is contained in the δ -neighborhood of the union of the two other sides. We denote by $\delta(T)$ the sharp thin constant of T , i.e. $\delta(T) := \inf\{\delta \geq 0 : T \text{ is } \delta\text{-thin}\}$. The space X is δ -hyperbolic (or satisfies the *Rips condition* with constant δ) if every geodesic triangle in X is δ -thin. We denote by $\delta(X)$ the sharp hyperbolicity constant of X , i.e. $\delta(X) := \sup\{\delta(T) : T \text{ is a geodesic triangle in } X\}$. We say that X is *hyperbolic* if X is δ -hyperbolic for some $\delta \geq 0$; then X is hyperbolic if and only if $\delta(X) < \infty$. If we have a triangle with two identical vertices, we call it a “bigon”. Obviously, every bigon in a δ -hyperbolic space is δ -thin.

Trivially, any bounded metric space X is $(\text{diam } X)$ -hyperbolic. A normed linear space is hyperbolic if and only if it has dimension one. If a complete Riemannian manifold is simply connected and its sectional curvatures satisfy $K \leq c$ for some negative constant c , then it is hyperbolic. See the classical references [1, 18] in order to find further results. We want to remark that the main examples of hyperbolic graphs are the trees. In fact, the hyperbolicity constant of a geodesic metric space can be viewed as a measure of how “tree-like” the space is, since those spaces X with $\delta(X) = 0$ are precisely the metric trees. This is an interesting subject since, in many applications, one finds that the borderline between tractable and intractable cases may be the tree-like degree of the structure to be dealt with (see, e.g., [13]).

An graph H is a *minor* of a graph G if a graph isomorphic to H can be obtained from G by contracting some edges, deleting some edges, and deleting some isolated vertices. Minor graphs is an interesting class of graphs. This topic started with one well-known result on planar graph, independently proved by Kuratowski and Wagner, which says that a graph is planar if and only if it do not include as a minor the complete graph K_5 nor the complete bipartite graph $K_{3,3}$ (see [25, 38]).

For a general graph or a general geodesic metric space deciding whether or not a space is hyperbolic is usually very difficult. Therefore, it is interesting to study the invariance of the hyperbolicity of graphs under appropriate transformations. The invariance of the hyperbolicity under some natural transformations on graphs have been studied in previous papers, for instance, removing edges of a graph is studied in [4, 7], the line graph of a graph in [9, 11], the dual of a planar graph in [8, 29] and the complement of a graph in [5].

To remove and to contract edges of a graph are also very natural transformations. In [7] the authors study the distortion of the hyperbolicity constant of the graph $G \setminus e$ obtained from a graph G by removing an edge e . These results allow to obtain the characterization, in a quantitative way, of the hyperbolicity of many graphs in terms of local hyperbolicity. In this work we obtain the invariance of the hyperbolicity under the contraction of a finite number of edges. Besides, we obtain quantitative information about the distortion of the hyperbolicity constant of the graph G/e obtained from the graph G by contracting an arbitrary edge e from it for simple and non-simple graphs, in Sections 2 and 4, respectively. In Sections 3 and 4 we obtain the invariance of the hyperbolicity on many minor graphs as a consequence of these results for simple and non-simple graphs, respectively.

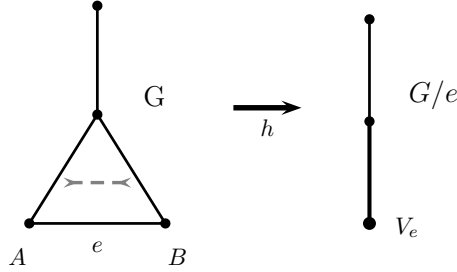
2. HYPERBOLICITY AND EDGE CONTRACTION ON SIMPLE GRAPHS

In this section we study the distortion of the hyperbolicity constant by contraction of one edge in any simple graph. If G is a graph and $e := [A, B] \in E(G)$, we denote by G/e the graph obtained by contracting the edge e from it (we remove e from G while simultaneously we merge A and B).

Along this work we will denote by V_e the vertex in G/e obtained by identifying A and B .

Note that any vertex $v \in V(G) \setminus \{A, B\}$ can be seen as a vertex in $V(G/e)$. Also we can write any edge in $E(G/e)$ in terms of its endpoints, but we write V_e instead of A or B . If $[v, A]$ and $[v, B]$ are edges of G for some $v \in V(G)$, then we replace both edges by a single edge $[v, V_e] \in G/e$ (recall that we work with simple graphs), see Figure 1.

We define the map $h : G \rightarrow G/e$ in the following way: if x belongs to the edge e , then $h(x) := V_e$; if $x \in G$ does not belong to e , then $h(x)$ is the “natural inclusion map”. Clearly h is onto, *i.e.*, $h(G) = G/e$. Besides, h is an injective map in the union of edges without endpoints in $\{A, B\}$.

FIGURE 1. The map h .

Denote by $J(G)$ the set of vertices and midpoints of edges in G .

As usual, by *cycle* we mean a simple closed curve, *i.e.*, a path with different vertices, unless the last one, which is equal to the first vertex.

Given $e \in E(G)$, denote by $\mathcal{C}(G, e)$ the set of cycles in G with length 3 containing e .

Proposition 2.1. *Let G be a graph and $e \in E(G)$. Then*

$$(2.1) \quad d_{G/e}(h(x), h(y)) \leq d_G(x, y) \leq d_{G/e}(h(x), h(y)) + \frac{3}{2}, \quad \forall x, y \in G.$$

Furthermore, if $y \in J(G)$ or x, y are not contained in the same cycle $C \in \mathcal{C}(G, e)$, then

$$(2.2) \quad d_{G/e}(h(x), h(y)) \leq d_G(x, y) \leq d_{G/e}(h(x), h(y)) + 1.$$

Proof. Fix $x, y \in G$. Let $[xy]_G$ be a geodesic in G joining x and y . Clearly, $h([xy]_G)$ is a path joining $h(x)$ and $h(y)$ with length at most $L([xy]_G)$, thus, we obtain $d_{G/e}(h(x), h(y)) \leq d_G(x, y)$. Hence, the first inequalities in (2.1) and (2.2) hold.

Let γ' be a geodesic in G/e joining $h(x)$ and $h(y)$. If x, y are not contained in the same cycle $C \in \mathcal{C}(G, e)$, then there is a path γ in G with $h(\gamma) = \gamma'$ and $L(\gamma) \leq L(\gamma') + 1$ since e (or a subset of e) can be included in γ . Therefore, $d_G(x, y) \leq L(\gamma) \leq L(\gamma') + 1 = d_{G/e}(h(x), h(y)) + 1$.

If $x, y \in C \in \mathcal{C}(G, e)$, then

$$d_G(x, y) \leq d_C(x, y) \leq \frac{1}{2} L(C) = \frac{3}{2} \leq d_{G/e}(h(x), h(y)) + \frac{3}{2}.$$

Finally, consider $x, y \in C \in \mathcal{C}(G, e)$ with $y \in J(G)$.

We deal with several cases.

Case 1. If $d_{G/e}(h(x), h(y)) \geq 1/2$, then

$$d_G(x, y) \leq \frac{1}{2} L(C) = \frac{1}{2} + 1 \leq d_{G/e}(h(x), h(y)) + 1.$$

Case 2. Assume that $d_{G/e}(h(x), h(y)) < 1/2$.

Case 2.1. y is the midpoint of e . If $x \in e$, then $d_G(x, y) \leq 1/2$. If $x \notin e$, then $d_G(x, y) = d_{G/e}(h(x), h(y)) + 1/2$.

Case 2.2. $y \in \{A, B\}$. If $x \in e$, then $d_G(x, y) \leq 1$. If $x \notin e$, then we have either $d_G(x, y) = d_{G/e}(h(x), h(y))$ or $d_G(x, y) = d_{G/e}(h(x), h(y)) + 1$.

Case 2.3. $y \in V(C) \setminus \{A, B\}$. Thus $d_G(x, y) = d_{G/e}(h(x), h(y))$.

Case 2.4. $y \in J(G) \setminus \{V(C) \cup e\}$. We have either $d_G(x, y) = d_{G/e}(h(x), h(y))$, $d_G(x, y) = d_{G/e}(h(x), h(y)) + 1$ or $d_G(x, y) = 1 - d_{G/e}(h(x), h(y)) \leq 1$.

This finishes the proof. \square

Note that the inequalities in (2.1) and (2.2) are attained. If G is any graph, $[v, w] \in E(G)$ and $\{v, w\} \cap \{A, B\} = \emptyset$, then $d_{G/e}(h(v), h(w)) = 1 = d_G(v, w)$. Consider a cycle graph $G = C_3$ and $x, y \in C_3$ such that $x \neq y$ and there is $v \in V(C_3)$ with $d_{C_3}(x, v) = d_{C_3}(v, y) = 3/4$. Let e be the edge in C_3 with $x, y \notin e$. Hence, we have $d_{C_3}(x, y) = 3/2$ and $h(x) = h(y)$. Finally, consider a cycle graph $G = C_3$, x_0, y_0 two different midpoints of edges in C_3 and $e \in E(C_3)$ with $x_0, y_0 \notin e$. Thus, $d_{C_3}(x_0, y_0) = 1$ and $h(x_0) = h(y_0)$.

The previous lemma has the following consequence about the continuity of h .

Proposition 2.2. *The map h is an 1-Lipschitz continuous function.*

Let (X, d_X) and (Y, d_Y) be two metric spaces. A map $f : X \rightarrow Y$ is said to be an (α, β) -quasi-isometric embedding, with constants $\alpha \geq 1$, $\beta \geq 0$ if for every $x, y \in X$:

$$\alpha^{-1}d_X(x, y) - \beta \leq d_Y(f(x), f(y)) \leq \alpha d_X(x, y) + \beta.$$

The function f is ε -full if for each $y \in Y$ there exists $x \in X$ with $d_Y(f(x), y) \leq \varepsilon$.

A map $f : X \rightarrow Y$ is said to be a quasi-isometry if there exist constants $\alpha \geq 1$, $\beta, \varepsilon \geq 0$ such that f is an ε -full (α, β) -quasi-isometric embedding.

A fundamental property of hyperbolic spaces is the following:

Theorem 2.3 (Invariance of hyperbolicity). *Let $f : X \rightarrow Y$ be an (α, β) -quasi-isometric embedding between the geodesic metric spaces X and Y . If Y is hyperbolic, then X is hyperbolic. Furthermore, if Y is δ -hyperbolic, then X is δ' -hyperbolic, where δ' is a constant which just depends on α, β, δ .*

Besides, if f is ε -full for some $\varepsilon \geq 0$ (a quasi-isometry), then X is hyperbolic if and only if Y is hyperbolic. Furthermore, if X is δ -hyperbolic, then Y is δ' -hyperbolic, where δ' is a constant which just depends on $\alpha, \beta, \delta, \varepsilon$.

Using the invariance of hyperbolicity (Theorem 2.3), we can obtain the main qualitative result in this section.

Theorem 2.4. *Let G be a graph and $e \in E(G)$. Then G is hyperbolic if and only if G/e is hyperbolic. Furthermore, if G (respectively, G/e) is δ -hyperbolic, then G/e (respectively, G) is δ' -hyperbolic, where δ' is a constant which just depends on δ .*

Proof. Proposition 2.1 gives that h is a 0-full $(1, 3/2)$ -quasi-isometry from G onto G/e , and we obtain the result by Theorem 2.3. \square

One can expect that the edge contraction is a monotone transformation for the hyperbolicity constant, i.e., the hyperbolicity constant always decreases by edge contraction (for instance, if e is any edge of the cycle graph C_3 , then C_3/e is the path graph P_2 and $\delta(P_2) = 0 < 3/4 = \delta(C_3)$; if e is any edge of the cycle graph C_n with $n \geq 4$, then C_n/e is the cycle graph C_{n-1} and $\delta(C_{n-1}) = (n-1)/4 < n/4 = \delta(C_n)$).

However, the following example provides a family of graphs such that the hyperbolicity constant increases by contracting certain edge. Recall that the *girth* of a graph G is the minimum of the lengths of the cycles in G .

Example 2.5. In [32, Theorem 11], the authors obtain the precise value of the hyperbolicity constant of the wheel graph with n vertices W_n : $\delta(W_4) = \delta(W_5) = 1$, $\delta(W_n) = 3/2$ for every $7 \leq n \leq 10$, and $\delta(W_n) = 5/4$ for $n = 6$ and for every $n \geq 11$. Note that we can obtain W_n from W_{n+1} by edge contraction, so, we have that $\delta(W_{11}) = 5/4$ and $\delta(W_{10}) = 3/2$. Furthermore, in [10] the authors obtain the value of the hyperbolicity constant of the graph join of E (the empty graph with just one vertex) and every graph. Thus, taking G as a graph join of E and any graph G^* with girth 10, then $\delta(G) = 5/4$, but contracting an edge e belonging to any cycle in G^* with length 10, G/e is the graph join of E and other graph with girth 9, so [10, Corollary 3.23] gives $\delta(G/e) = 3/2$.

Other aim in this work is to obtain quantitative relations between $\delta(G/e)$ and $\delta(G)$. Since the proofs of these inequalities are long, in order to make the arguments more transparent, we collect some results in technical lemmas.

For any simple (non-selfintersecting) path γ' joining two different points in G/e which are not contained in an edge e_0 with $h^{-1}(e_0) \in \mathcal{C}(G, e)$, we define $\Gamma(\gamma')$ as the set of paths γ in G such that $h(\gamma) = \gamma'$ and

$$\Gamma_0(\gamma') = \{g \in \Gamma(\gamma') \mid L(g) \leq L(\gamma) \quad \forall \gamma \in \Gamma(\gamma')\}.$$

Note that any curve in $\Gamma_0(\gamma')$ is a simple path. We denote by $h_0^{-1}(\gamma')$ any fixed choice of curve in $\Gamma_0(\gamma')$. If $t' \in \gamma' \setminus V_e$ we denote by $t = h_0^{-1}(t')$ the point in $h_0^{-1}(\gamma')$ such that $h(t) = t'$ (note that, since γ' is a simple path, any $t' \in \gamma' \setminus V_e$ defines a unique $t \in h_0^{-1}(\gamma')$). If $t' = V_e \in \gamma'$, then $h_0^{-1}(V_e) = h^{-1}(V_e) = e$. Hence, $h_0^{-1}(t') = h^{-1}(t') \cap h_0^{-1}(\gamma')$. Furthermore, for any geodesic γ' in G/e such that $V_e \notin \gamma'$ we have that

$$h|_{h^{-1}(\gamma')}: h^{-1}(\gamma') \longrightarrow \gamma'$$

is a bijective map and $\gamma = h^{-1}(\gamma') = h_0^{-1}(\gamma')$ is a geodesic in G with $L(\gamma) = L(\gamma')$.

Lemma 2.6. Let G be a graph and $e \in E(G)$. Let $x, y \in G \setminus \{e\}$ such that there is no $C \in \mathcal{C}(G, e)$ with $x, y \in C$. Assume that there are two geodesics γ_G and $\gamma_{G/e}$ in G and G/e , respectively, joining x, y and $h(x), h(y)$, respectively, such that $L(\gamma_G) = L(\gamma_{G/e}) = L(h(\gamma_G))$ and $e \subset h_0^{-1}(\gamma_{G/e})$. Then we have

$$(2.3) \quad d_{G/e}(h(\alpha), \gamma_{G/e}) \leq \delta(G) \quad \forall \alpha \in \gamma_G$$

and

$$(2.4) \quad d_{G/e}(\alpha', h(\gamma_G)) \leq 2\delta(G) \quad \forall \alpha' \in \gamma_{G/e}.$$

Remark 2.7. Since there is no $C \in \mathcal{C}(G, e)$ with $x, y \in C$, we deduce that $h(x), h(y)$ are not contained in an edge e_0 with $h^{-1}(e_0) \in \mathcal{C}(G, e)$, and then $h_0^{-1}(\gamma_{G/e})$ is well defined.

Proof. Without loss of generality we can assume that G is hyperbolic, since otherwise the inequalities trivially hold. Let z be the midpoint of $e = [A, B]$. By symmetry, we can assume that the closure of the connected components of $h_0^{-1}(\gamma_{G/e} \setminus \{V_e\})$ join x with A , and B with y . Clearly,

$$d_G(z, x) = d_G(z, A) + \frac{1}{2} = d_{G/e}(h(x), V_e) + \frac{1}{2}, \quad d_G(z, y) = d_G(y, B) + \frac{1}{2} = d_{G/e}(h(y), V_e) + \frac{1}{2}.$$

So, there are geodesics $[xz]_G$ and $[zy]_G$ in G verifying the following: $[xz]_G$ contains the closure of the connected component of $h_0^{-1}(\gamma_{G/e} \setminus \{V_e\})$ joining x with A , and $[zy]_G$ contains the closure of the connected component of $h_0^{-1}(\gamma_{G/e} \setminus \{V_e\})$ joining B with y . Hence, $T := \{\gamma_G, [yz]_G, [zx]_G\}$ is a geodesic triangle in G and so,

$$d_G(\alpha, [yz]_G \cup [zx]_G) = d_G(\alpha, h_0^{-1}(\gamma_{G/e})) \leq \delta(G)$$

for every $\alpha \in \gamma_G$, and (2.3) holds by Proposition 2.1.

In order to obtain (2.4), without loss of generality we can assume that $\alpha' \in h([yz]_G)$, since $\gamma_{G/e} = h([yz]_G) \cup h([zx]_G)$. Denote by α a point in $[yB]_G$ with $h(\alpha) = \alpha'$. If $d_G(y, e) = d_G(y, B) \leq 2\delta(G)$, then we have

$$d_{G/e}(\alpha', h(\gamma_G)) \leq d_G(\alpha, \gamma_G) \leq d_G(\alpha, y) \leq d_G(B, y) \leq 2\delta(G).$$

Assume that $d_G(y, e) > 2\delta(G)$. Now we can take a point $w \in [yz]_G$ such that $d_G(w, e) = \delta(G)$. If $\alpha \in [wy] \setminus \{w\}$, then the hyperbolicity of G implies $d_G(\alpha, \gamma_G \cup [zx]_G) \leq \delta(G)$; note that $d_G(\alpha, [zx]_G) > \delta(G)$ since $\gamma_{G/e}$ is a geodesic in G/e and $d_G(w, e) = \delta(G)$. Hence, $d_G(\alpha, \gamma_G) \leq \delta(G)$ and thus Proposition 2.1 gives $d_{G/e}(\alpha', h(\gamma_G)) \leq \delta(G)$. Assume now that $\alpha \in [wB] \setminus B$ (therefore, $\alpha' \neq V_e$). Thus, there exists $\alpha_1 \in [yz]_G$ such that $d_G(\alpha, \alpha_1) = \delta(G)$ and $d_G(\alpha_1, e) > \delta(G)$. Therefore, $\alpha_1 \in [wy] \setminus \{w\}$ and we have proved that $d_G(\alpha_1, \gamma_G) \leq \delta(G)$. Hence, Proposition 2.1 gives

$$d_{G/e}(\alpha', h(\gamma_G)) \leq d_G(\alpha, \gamma_G) \leq d_G(\alpha, \alpha_1) + d_G(\alpha_1, \gamma_G) \leq 2\delta(G).$$

If $\alpha = B$, then $\alpha' = V_e$ and the inequality for $\alpha' = V_e$ is obtained by continuity. \square

In what follows, if x, y belong to some $C \in \mathcal{C}(G, e)$, then we denote by $h_0^{-1}([h(x)h(y)]_{G/e})$ any fixed choice of a geodesic in C with $[h(x)h(y)]_{G/e} \subseteq h(h_0^{-1}([h(x)h(y)]_{G/e}))$, and by $h_0^{-1}(\alpha')$ any point in $h^{-1}(\alpha') \cap h_0^{-1}([h(x)h(y)]_{G/e})$.

Lemma 2.8. *Let G be a graph and $e \in E(G)$ such that G/e is not a tree. Let $[xy]_G$ be a geodesic in G joining $x, y \in J(G)$. Assume that $h([xy]_G)$ is not a geodesic in G/e and let $[h(x)h(y)]_{G/e}$ be a geodesic in G/e joining $h(x)$ and $h(y)$. Then we have*

$$(2.5) \quad d_G(h_0^{-1}(\alpha'), [xy]_G) \leq \delta(G/e) + 1 \leq \frac{7}{3}\delta(G/e), \quad \forall \alpha' \in [h(x)h(y)]_{G/e}$$

and

$$(2.6) \quad d_G(\alpha, h_0^{-1}([h(x)h(y)]_{G/e})) \leq 2\delta(G/e), \quad \forall \alpha \in [xy]_G.$$

Proof. Without loss of generality we can assume that G/e is hyperbolic, since otherwise the inequalities hold. Since G/e is not a tree, $\delta(G/e) \geq 3/4$ by [26, Theorem 11], and

$$\delta(G/e) + 1 \leq \frac{7}{3}\delta(G/e).$$

We deal with several cases.

(a) There exists $C \in \mathcal{C}(G, e)$ with $x, y \in C$. For every $\alpha, \beta \in C$ we have

$$d_G(\alpha, \beta) \leq \text{diam } C = \frac{3}{2} = \min \left\{ \frac{3}{4} + 1, 2\frac{3}{4} \right\} \leq \min \{ \delta(G/e) + 1, 2\delta(G/e) \}.$$

Since $[xy]_G$ and $h_0^{-1}([h(x)h(y)]_{G/e})$ are contained in C , (2.5) and (2.6) hold.

(b) There is no $C \in \mathcal{C}(G, e)$ with $x, y \in C$. Thus $h(x), h(y)$ are not contained in an edge e_0 with $h^{-1}(e_0) \in \mathcal{C}(G, e)$, and then $h_0^{-1}([h(x)h(y)]_{G/e})$ is defined as before Lemma 2.6.

Let $[A, B] := e$. Since $h([xy]_G)$ is not a geodesic in G/e and $x, y \in J(G)$, we have $e \cap [xy]_G \subsetneq \{A, B\}$, $L([xy]_G) \geq 3/2$, $d_G(x, y) = d_{G/e}(h(x), h(y)) + 1$ and $V_e \in [h(x)h(y)]_{G/e}$.

(b.1) Assume that there exists a cycle $C \in \mathcal{C}(G, e)$ with $L([xy]_G \cap C) > 1$. Since $[xy]_G$ is a geodesic in G , $h([xy]_G)$ is not a geodesic in G/e and $x, y \in J(G)$, we have $L([xy]_G \cap C) = 3/2$, $C \subset h_0^{-1}([h(x)h(y)]_{G/e}) \cup [xy]_G$, the closures of $h_0^{-1}([h(x)h(y)]_{G/e}) \setminus C$ and $[xy]_G \setminus C$ are two geodesics in G with the same endpoints, and the closures of $[h(x)h(y)]_{G/e} \setminus h(C)$ and $h([xy]_G) \setminus h(C)$ are two geodesics in G/e with the same endpoints. Since $L([xy]_G \cap C) = 3/2 = L(h_0^{-1}([h(x)h(y)]_{G/e}) \cap C)$ and $L(C) = 3$, we have

$$\begin{aligned} d_G(h_0^{-1}(\alpha'), [xy]_G) &\leq 3/4 \leq \delta(G/e), \quad \forall \alpha' \in [h(x)h(y)]_{G/e} \cap h(C), \\ d_G(\alpha, h_0^{-1}([h(x)h(y)]_{G/e})) &\leq 3/4 \leq \delta(G/e), \quad \forall \alpha \in [xy]_G \cap C. \end{aligned}$$

Since the closures of $h_0^{-1}([h(x)h(y)]_{G/e}) \setminus C$ and $[xy]_G \setminus C$ are two geodesics in G with the same endpoints, and the closures of $[h(x)h(y)]_{G/e} \setminus h(C)$ and $h([xy]_G) \setminus h(C)$ are two geodesics in G/e with the same endpoints, we also have

$$d_G(h_0^{-1}(\alpha'), [xy]_G) \leq d_G(h_0^{-1}(\alpha'), [xy]_G \setminus C) = d_{G/e}(\alpha', h([xy]_G) \setminus h(C)) \leq \delta(G/e),$$

for every $\alpha' \in [h(x)h(y)]_{G/e} \setminus h(C)$, and

$$d_G(\alpha, h_0^{-1}([h(x)h(y)]_{G/e})) \leq d_G(\alpha, h_0^{-1}([h(x)h(y)]_{G/e}) \setminus C) = d_{G/e}(h(\alpha), [h(x)h(y)]_{G/e} \setminus h(C)) \leq \delta(G/e),$$

for every $\alpha \in [xy]_G \setminus C$.

(b.2) Assume now that $L([xy]_G \cap C) \leq 1$ for every $C \in \mathcal{C}(G, e)$.

Note that $L(h([xy]_G)) = L([xy]_G)$, since $e \cap [xy]_G \subset \{A, B\}$, so, for any $z \in [xy]_G$ we have $L(h([xz]_G)) = L([xz]_G)$ and $L(h([zy]_G)) = L([zy]_G)$. Consider the points $A', B' \in [xy]_G$ such that $d_G(x, A') = d_G(x, e)$ and $d_G(y, B') = d_G(y, e)$. Since $x, y \in J(G)$, we have $A', B' \in V(G)$. Since $L(h([xy]_G)) = L([xy]_G)$, $[A', B'] \in E(G)$ and $[A', B'] \subset [xy]_G$. Let z be the midpoint of $[A', B']$. Since $d(x, A') = d(x, e)$, $d_G(y, B') = d_G(y, e)$, $d_G(z, A') = d_G(z, B') = 1/2$ and $L([xy]_G \cap C) \leq 1$ for every $C \in \mathcal{C}(G, e)$, we have that $h([xz]_G)$ and $h([zy]_G)$ are geodesics in G/e . Hence, $T = \{[h(x)h(y)]_{G/e}, h([yz]_G), h([zx]_G)\}$ is a geodesic triangle in G/e , and thus

$$\begin{aligned} d_G(h_0^{-1}(\alpha'), [xy]_G) &\leq d_{G/e}(\alpha', h([xy]_G)) + 1 = d_{G/e}(\alpha', h([xz]_G) \cup h([zy]_G)) + 1 \\ &\leq \delta(T) + 1 \leq \delta(G/e) + 1 \leq \frac{7}{3} \delta(G/e), \end{aligned}$$

for every $\alpha' \in [h(x)h(y)]_{G/e}$.

In order to obtain (2.6), without loss of generality we can assume that $\alpha \in [yz]_G$. If $L([yz]_G) \leq 2\delta(G/e)$, then we have $d_G(\alpha, h_0^{-1}([h(x)h(y)]_{G/e})) \leq d_G(\alpha, y) \leq 2\delta(G/e)$. Assume that $L([yz]_G) > 2\delta(G/e)$. Let w be the point in $[yz]_G$ with $d_G(w, z) = \delta(G/e)$. If $\alpha \in [wy]_G \setminus \{w\}$, then the hyperbolicity of G/e implies $d_{G/e}(h(\alpha), [h(x)h(y)]_{G/e} \cup h([zx]_G)) \leq \delta(G/e)$. Note that if $d_{G/e}(h(\alpha), h([zx]_G)) \leq \delta(G/e)$, then a geodesic γ joining $h(\alpha)$ and $h([zx]_G)$ in G/e contains V_e and, since $V_e \in [h(x)h(y)]_{G/e}$, we obtain

$$d_{G/e}(h(\alpha), [h(x)h(y)]_{G/e}) \leq d_{G/e}(h(\alpha), V_e) \leq L(\gamma) \leq \delta(G/e).$$

Thus, we have $d_{G/e}(h(\alpha), [h(x)h(y)]_{G/e}) \leq \delta(G/e)$. Hence, we obtain

$$d_G(\alpha, h_0^{-1}([h(x)h(y)]_{G/e})) = d_{G/e}(h(\alpha), [h(x)h(y)]_{G/e}) \leq \delta(G/e).$$

Assume now that $\alpha \in [zw]_G \setminus \{z\}$. Then, there exists $\alpha_1 \in [wy]_G \setminus \{w\}$ such that $d_G(\alpha, \alpha_1) = \delta(G/e)$, and we deduce

$$d_G(\alpha, h_0^{-1}([h(x)h(y)]_{G/e})) \leq d_G(\alpha, \alpha_1) + d_G(\alpha_1, h_0^{-1}([h(x)h(y)]_{G/e})) \leq 2\delta(G/e).$$

The inequality for $\alpha = z$ is obtained by continuity. \square

Remark 2.9. Let G be a graph, $e \in E(G)$ and $T = \{\gamma_1, \gamma_2, \gamma_3\}$ a geodesic triangle in G . Then at least one of the curves $h(\gamma_1), h(\gamma_2), h(\gamma_3)$ is a geodesic in G/e , since otherwise there exists another geodesic triangle $T' = \{\gamma'_1, \gamma'_2, \gamma'_3\}$ with the same vertices that T and such that the edge e is contained in $\gamma'_1 \cap \gamma'_2 \cap \gamma'_3$.

The following result will be useful.

Theorem 2.10. [3, Theorem 2.7] For a hyperbolic graph G , there exists a geodesic triangle $T = \{x, y, z\}$ that is a cycle with $x, y, z \in J(G)$ and $\delta(T) = \delta(G)$.

In order to prove Theorem 2.12 we will need the following technical result.

Lemma 2.11. Let G be a graph and $e \in E(G)$. If G/e is a tree, then

$$\delta(G) \leq 1.$$

Proof. If G is a tree, then $\delta(G) = 0 \leq 1$.

Assume now that G is not a tree. Since G/e is a tree, if an edge e_0 is contained in a cycle in G , then it is contained in some cycle $C_0 \in \mathcal{C}(G, e)$ and it contains A or B . Since any cycle in $\mathcal{C}(G, e)$ contains the edge e , if $[A, B] = e$, then every cycle in G contains the vertices A and B . Therefore, any cycle in G has length at most 4.

Theorem 2.4 gives that G is hyperbolic since $\delta(G/e) = 0$. Hence, by Theorem 2.10 there exist a geodesic triangle $T = \{x, y, z\}$ in G that is a cycle with $x, y, z \in J(G)$ and $p \in [xy]$ with $d_G(p, [yz] \cup [zx]) = \delta(T) = \delta(G)$. Since T is a cycle, we have seen that $L(T) \leq 4$. Thus

$$\delta(G) = d_G(p, [yz] \cup [zx]) \leq d_G(p, \{x, y\}) \leq \frac{1}{2} d_G(x, y) \leq \frac{1}{4} L(T) \leq 1.$$

□

The previous results allow to obtain a quantitative version of Theorem 2.4.

Theorem 2.12. *Let G be a graph and $e \in E(G)$. Then*

$$(2.7) \quad \frac{1}{3} \delta(G/e) \leq \delta(G) \leq \frac{16}{3} \delta(G/e) + 1.$$

Proof. By Theorem 2.4 we have that G and G/e are hyperbolic or not simultaneously. If G and G/e are not hyperbolic, then $\delta(G) = \delta(G/e) = \infty$ and (2.7) holds. Assume now that both graphs are hyperbolic.

Let us prove the first inequality in (2.7). If G is a tree, then $\delta(G/e) = 0$ and the first inequality holds. Assume that G/e is not a tree, thus $\delta(G/e) > 0$. By Theorem 2.10 there exist a geodesic triangle $T' = \{[x'y'], [y'z'], [z'x']\}$ in G/e that is a cycle with $x', y', z' \in J(G/e)$ and $p' \in [x'y']$ with $d_{G/e}(p', [y'z'] \cup [z'x']) = \delta(T') = \delta(G/e)$.

Consider $T \subset h^{-1}([x'y'] \cup [y'z'] \cup [z'x'])$ such that T is a cycle with $h(T) = T'$. Define $x := h^{-1}(x') \cap T$, $y := h^{-1}(y') \cap T$ and $z := h^{-1}(z') \cap T$, if $V_e \notin \{x', y', z'\}$; otherwise, if $V_e = a'$ with $a' \in \{x', y', z'\}$, then we define a as the midpoint of e . Hence, we can define g_{ab} as the simple curve contained in T joining a and b , and such that $h(g_{ab}) = [a'b']$, for $a, b \in \{x, y, z\}$ (note that $g_{ab} = h_0^{-1}([a'b'])$ if $h_0^{-1}([a'b'])$ is defined as before Lemma 2.6, i.e., if a', b' are not contained in an edge e_0 with $h_0^{-1}(e_0) \in \mathcal{C}(G, e)$). Then $x, y, z \in J(G)$ and T can be seen as the triangle $\{g_{xy}, g_{yz}, g_{zx}\}$. Note that if $V_e \in \{x', y', z'\}$, then T is a geodesic triangle in G .

We deal with several cases.

(a) If T is a geodesic triangle in G , then by Proposition 2.1 we have for any $p \in h^{-1}(p') \cap g_{xy}$

$$\delta(G/e) = \delta(T') = d_{G/e}(p', [y'z'] \cup [z'x']) \leq d_G(p, g_{yz} \cup g_{zx}) \leq \delta(T) \leq \delta(G)$$

and so, the first inequality in (2.7) holds.

(b) Assume that T is not a geodesic triangle in G . Thus, we have $V_e \in T' \setminus \{x', y', z'\}$, $e \subset T$ and $L(T) = L(T') + 1$. Since $V_e \notin \{x', y', z'\}$, the edge e is contained in exactly one of g_{xy}, g_{yz}, g_{zx} , and the other two paths are geodesics in G by Proposition 2.1.

(b.1) Assume that $e \subset g_{xy}$. Note that e is contained in the interior of g_{xy} (recall that $V_e \notin \{x', y', z'\}$); since $x, y \in J(G)$, we have $L(g_{xy}) \geq 2$. Therefore, $x, y \in G \setminus \{e\}$ and there is no $C \in \mathcal{C}(G, e)$ with $x, y \in C$. Hence, $h_0^{-1}([x'y'])$ is defined as before Lemma 2.6, $g_{xy} = h_0^{-1}([x'y'])$ and $e \subset h_0^{-1}([x'y'])$. Consider a geodesic $[xy]$ in G joining x and y . We have $L([xy]) = L([x'y']) = L(h([xy]))$. Note that g_{xy} is not a geodesic by hypothesis. By Lemma 2.6 there is $p \in [xy]$ such that $d_{G/e}(p', h(p)) \leq 2\delta(G)$. Thus, since $\{[xy], g_{yz}, g_{zx}\}$ is a geodesic triangle in G , there is $p_1 \in g_{yz} \cup g_{zx}$ such that $d_G(p, p_1) \leq \delta(G)$. Hence, Proposition 2.1 gives

$$\begin{aligned} \delta(G/e) &= d_{G/e}(p', [y'z'] \cup [z'x']) \leq \delta_{G/e}(p', h(p_1)) \leq d_{G/e}(p', h(p)) + d_{G/e}(h(p), h(p_1)) \\ &\leq 2\delta(G) + d_G(p, p_1) \leq 3\delta(G). \end{aligned}$$

(b.2) Assume now that $e \subset g_{yz} \cup g_{zx}$. By symmetry, we can assume that $e \subset g_{yz}$. Note that g_{yz} is not a geodesic by hypothesis, and that g_{xy}, g_{zx} are geodesics. Consider a geodesic $[yz]$ in G joining y and z . Since $\{g_{xy}, [yz], g_{zx}\}$ is a geodesic triangle in G , there is $p \in [yz] \cup g_{zx}$ such that $d_G(h_0^{-1}(p'), p) \leq \delta(G)$.

(b.2.1) If $p \in g_{zx}$, then Proposition 2.1 gives

$$\delta(G/e) = d_{G/e}(p', [y'z'] \cup [z'x']) \leq d_{G/e}(p', h(p)) \leq d_G(h_0^{-1}(p'), p) \leq \delta(G).$$

(b.2.2) Assume that $p \in [yz]$. The argument in (b.1) also gives that $y, z \in G \setminus \{e\}$, there is no $C \in \mathcal{C}(G, e)$ with $y, z \in C$, $e \subset g_{yz} = h_0^{-1}([y'z'])$, and $L([yz]) = L([y'z']) = L(h([yz]))$. Therefore, by Lemma 2.6 there is $p'_1 \in [y'z']$ such that $d_{G/e}(h(p), p'_1) \leq \delta(G)$. Thus, we have by Proposition 2.1

$$\begin{aligned} \delta(G/e) &= d_{G/e}(p', [y'z'] \cup [z'x']) \leq d_{G/e}(p', p'_1) \leq d_{G/e}(p', h(p)) + d_{G/e}(h(p), p'_1) \\ &\leq d_G(h_0^{-1}(p'), p) + \delta(G) \leq 2\delta(G). \end{aligned}$$

Hence, the first inequality in (2.7) holds.

Let us prove the second inequality in (2.7). By Theorem 2.10 there exist a geodesic triangle $T = \{x, y, z\}$ in G that is a cycle with $x, y, z \in J(G)$ and $p \in [xy]$ with $d_G(p, [yz] \cup [zx]) = \delta(T) = \delta(G)$. Since $x, y, z \in J(G)$ we have

$$d_G(p, [yz] \cup [zx]) = d_G(p, J(G) \cap ([yz] \cup [zx])),$$

and if $d_G(p, [yz] \cup [zx]) = d_G(p, q)$ with $q \in J(G) \cap ([yz] \cup [zx])$, then Proposition 2.1 gives

$$d_G(p, [yz] \cup [zx]) = d_G(p, q) \leq d_{G/e}(h(p), h(q)) + 1.$$

If $\delta(G) \leq 1$, then the second inequality in (2.7) holds. Hence, we can assume that $\delta(G) > 1$. Note that since $\delta(G) > 1$ we have that G/e is not a tree by Lemma 2.11.

Let n be the number of geodesics in G/e of the set $\{h([xy]), h([yz]), h([zx])\}$. By Remark 2.9 we have $n \in \{1, 2, 3\}$.

We consider several cases.

(A) Assume that $h([xy])$ is a geodesic in G/e .

(A.1) If $n = 3$, then Proposition 2.1 gives

$$\delta(G) = d_G(p, [yz] \cup [zx]) \leq d_{G/e}(h(p), h([yz]) \cup h([zx])) + 1 \leq \delta(G/e) + 1.$$

(A.2) Consider the case $n = 2$. By symmetry we can assume that $h([yz])$ is a geodesic in G/e and let $[h(z)h(x)]$ be a geodesic in G/e joining $h(z)$ and $h(x)$. Then there is $p' \in h([yz]) \cup [h(z)h(x)]$ with $d_{G/e}(h(p), p') \leq \delta(G/e)$. By Proposition 2.1, we have $d_G(p, h_0^{-1}(p')) \leq d_{G/e}(h(p), p') + 1 \leq \delta(G/e) + 1$. If $p' \in h([yz])$, then $h_0^{-1}(p') \in [yz]$ and

$$\delta(G) = d_G(p, [yz] \cup [zx]) \leq d_G(p, h_0^{-1}(p')) \leq \delta(G/e) + 1.$$

Assume that $p' \in [h(z)h(x)]$. By Lemma 2.8, there is $p_1 \in [zx]$ with $d_G(h_0^{-1}(p'), p_1) \leq (7/3)\delta(G/e)$. Then we have

$$\delta(G) = d_G(p, [yz] \cup [zx]) \leq d_G(p, p_1) \leq d_G(p, h_0^{-1}(p')) + d_G(h_0^{-1}(p'), p_1) \leq \frac{10}{3}\delta(G/e) + 1.$$

(A.3) If $n = 1$, then let $[h(y)h(z)], [h(z)h(x)]$ be geodesics in G/e joining $h(y), h(z)$ and $h(z), h(x)$, respectively. Then there is $p' \in [h(y)h(z)] \cup [h(z)h(x)]$ with $d_{G/e}(h(p), p') \leq \delta(G/e)$. By Proposition 2.1 we have $d_G(p, h_0^{-1}(p')) \leq d_{G/e}(h(p), p') + 1 \leq \delta(G/e) + 1$. By symmetry we can assume that $p' \in [h(y)h(z)]$. By Lemma 2.8 there is $p_1 \in [yz]$ with $d_G(h_0^{-1}(p'), p_1) \leq (7/3)\delta(G/e)$. Thus,

$$\delta(G) = d_G(p, [yz] \cup [zx]) \leq d_G(p, p_1) \leq d_G(p, h_0^{-1}(p')) + d_G(h_0^{-1}(p'), p_1) \leq \frac{10}{3}\delta(G/e) + 1.$$

(B) Assume now that $h([xy])$ is not a geodesic in G/e . Let $[h(x)h(y)]$ be a geodesic in G/e joining $h(x)$ and $h(y)$. By Lemma 2.8 there is $p' \in [h(x)h(y)]$ with $d_G(p, h_0^{-1}(p')) \leq 2\delta(G/e)$.

(B.1) Consider the case $n = 2$, i.e., $h([yz])$ and $h([zx])$ are geodesics in G/e . Then there is $p'' \in h([yz]) \cup h([zx])$ with $d_{G/e}(p', p'') \leq \delta(G/e)$. By Proposition 2.1, we have

$$d_G(h_0^{-1}(p'), h_0^{-1}(p'')) \leq d_{G/e}(p', p'') + 1 \leq \delta(G/e) + 1.$$

Thus, $h_0^{-1}(p'') \in [yz] \cup [zx]$ and

$$\delta(G) = d_G(p, [yz] \cup [zx]) \leq d_G(p, h_0^{-1}(p'')) \leq d_G(p, h_0^{-1}(p')) + d_G(h_0^{-1}(p'), h_0^{-1}(p'')) \leq 3\delta(G/e) + 1.$$

(B.2) Consider the case $n = 1$. By symmetry we can assume that $h([yz])$ is a geodesic in G/e and let $[h(z)h(x)]$ be a geodesic in G/e joining $h(z)$ and $h(x)$. Then there is $p'' \in h([yz]) \cup [h(z)h(x)]$ with $d_{G/e}(p', p'') \leq \delta(G/e)$. By Proposition 2.1, we have

$$d_G(h_0^{-1}(p'), h_0^{-1}(p'')) \leq d_{G/e}(p', p'') + 1 \leq \delta(G/e) + 1.$$

If $p'' \in h([yz])$, then we obtain

$$\delta(G) = d_G(p, [yz] \cup [zx]) \leq d_G(p, h_0^{-1}(p'')) \leq d_G(p, h_0^{-1}(p')) + d_G(h_0^{-1}(p'), h_0^{-1}(p'')) \leq 3\delta(G/e) + 1.$$

If $p'' \in [h(z)h(x)]$, then by Lemma 2.8 there is $p_1 \in [zx]$ with $d_G(h_0^{-1}(p''), p_1) \leq (7/3)\delta(G/e)$. Then we have

$$\begin{aligned} \delta(G) &= d_G(p, [yz] \cup [zx]) \leq d_G(p, p_1) \\ &\leq d_G(p, h_0^{-1}(p')) + d_G(h_0^{-1}(p'), h_0^{-1}(p'')) + d_G(h_0^{-1}(p''), p_1) \\ &\leq \frac{16}{3}\delta(G/e) + 1. \end{aligned}$$

□

The bounds in Theorem 2.12 are sharp, as the following examples show.

Example 2.13. Let G_0 be the diamond graph, i.e., the complete graph with 4 vertices K_4 without one edge (see Figure 2). Let e be the edge joining the two vertices with degree 3 in G_0 . Then G_0/e is isomorphic to the path graph with 3 vertices P_3 . Clearly, we have $\delta(G_0) = 1$ and $\delta(G_0/e) = 0$. This fact allows to obtain many graphs G attaining the upper bound in Theorem 2.12: Consider any tree T and fix vertices $v \in V(T)$ and $u \in V(G_0 \setminus \{e\})$. Let G be the graph obtained from G_0 and T by identifying the vertices u and v . Then $\delta(G) = \delta(G_0) = 1$ and $\delta(G/e) = \delta(G_0/e) = 0$, since G/e and G_0/e are trees.

Example 2.14. If T is any tree and e is any edge of T , then T/e is also a tree, $\delta(T) = \delta(T/e) = 0$, and so, the lower bound in Theorem 2.12 is attained.

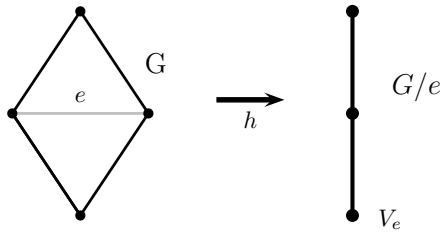


FIGURE 2. An example for which the upper bound in Theorem 2.12 is attained ($\delta(G) = 1$ and $\delta(G/e) = 0$).

Theorems 2.16 and 2.19 below allow to improve Theorem 2.12 in some special cases. In order to do it we need some previous result.

We say that a vertex v in a graph G is a *cut-vertex* if $G \setminus v$ is not connected. We say that an edge $e \in E(G)$ is a *cut-edge* if $G \setminus e$ is not connected. A graph is *two-connected* if it is connected and it does not contain cut-vertices. Given a graph G , we say that a family of subgraphs $\{G_s\}_s$ of G is a *T-decomposition* of G if $\cup_s G_s = G$ and $G_s \cap G_r$ is either a *cut-vertex* or the empty set for each $s \neq r$. Every graph has

a T-decomposition, as the following example shows. Given any edge in G , let us consider the maximal two-connected subgraph containing it. We call to the set of these maximal two-connected subgraphs $\{G_s\}_s$ the *canonical T-decomposition* of G .

In [4] the authors obtain the following result about T-decompositions.

Theorem 2.15. [4, Theorem 3] *Let G be a graph and $\{G_s\}_s$ be any T-decomposition of G , then*

$$\delta(G) = \sup_s \delta(G_s).$$

The following result improves Theorem 2.12 when e is a cut-edge.

Theorem 2.16. *Let G be a graph and e a cut-edge in G . Then*

$$\delta(G/e) = \delta(G).$$

Proof. Consider the connected components $\{G_s\}_s$ of $G \setminus e$. Then $\{G_s\}_s \cup \{e\}$ is a T-decomposition of G and Theorem 2.15 gives

$$\delta(G) = \max \left\{ \sup_s \delta(G_s), \delta(e) \right\} = \max \left\{ \sup_s \delta(G_s), 0 \right\} = \sup_s \delta(G_s).$$

For each s , let G'_s be the subgraph of G/e obtained from G_s by replacing the vertex in $\{A, B\}$ by V_e . Note that G'_s and G_s are isomorphic (and isometric) and, therefore, $\delta(G'_s) = \delta(G_s)$. Since $\{G'_s\}_s$ is a T-decomposition of G/e , Theorem 2.15 gives

$$\delta(G/e) = \sup_s \delta(G'_s) = \sup_s \delta(G_s) = \delta(G).$$

□

We say that a graph *has proper cycles* if each edge belongs at most to a cycle.

The *circumference* $c(G)$ of the graph G is the supremum of the lengths of cycles in G .

We need the following result.

Lemma 2.17. [33, Lemma 2.1] *In any graph G ,*

$$\delta(G) = \sup \left\{ \delta(T) : T \text{ is a geodesic triangle that is a cycle} \right\}.$$

Proposition 2.18. *Let G be a graph with proper cycles. Then*

$$\delta(G) = \frac{1}{4}c(G).$$

Proof. In order to bound $\delta(G)$, by Lemma 2.17 it suffices to consider geodesic triangles $T = \{x, y, z\}$ that are cycles. Hence, $L(T) \leq c(G)$, $d_G(x, y) \leq L(T)/2 \leq c(G)/2$ and $d_G(p, [yz] \cup [zx]) \leq d_G(p, \{x, y\}) \leq d_G(x, y)/2 \leq c(G)/4$ for every $p \in [xy]$. Therefore, $\delta(G) \leq c(G)/4$. Consider any fixed cycle C in G . Since G has proper cycles, $d_C(x, y) = d_G(x, y)$ for every $x, y \in C$. Choose $x, y \in C$ with $d_C(x, y) = L(C)/2$ and denote by g_1, g_2 the geodesics in C joining x and y with $g_1 \cup g_2 = C$ and $g_1 \cap g_2 = \{x, y\}$. Denote by B the geodesic bigon $B = \{g_1, g_2\}$. If p is the midpoint in g_1 , then $\delta(B) \geq d_G(p, g_2) = d_G(p, \{x, y\}) = d_G(x, y)/2 = L(C)/4$. Thus $\delta(G) \geq c(G)/4$, and we conclude $\delta(G) = c(G)/4$. □

We denote by $\mathfrak{C}(G)$ the set of cycles C in G with $L(C) = c(G)$ if $c(G) < \infty$. The following result improves Theorem 2.12 for graphs with proper cycles.

Theorem 2.19. *Let G be a graph with proper cycles.*

- *If either $c(G) = \infty$ or $\mathfrak{C}(G)$ contains at least two cycles, then $\delta(G/e) = \delta(G)$ for every $e \in E(G)$.*
- *If $\mathfrak{C}(G)$ contains just a single cycle C , then $\delta(G/e) = \delta(G)$ if and only if $e \notin E(C)$.*

Proof. Note that G/e also has proper cycles for every $e \in E(G)$ and $c(G/e)$ is equal to either $c(G)$ or $c(G) - 1$. Assume first that either $c(G) = \infty$ or $\mathfrak{C}(G)$ contains at least two cycles, and fix any $e \in E(G)$. Thus, $c(G/e) = c(G)$ and Proposition 2.18 gives $\delta(G/e) = \delta(G)$. Assume now that $\mathfrak{C}(G)$ contains just a single cycle C . If $e \notin E(C)$, then $c(G/e) = c(G)$ and $\delta(G/e) = \delta(G)$. If $e \in E(C)$, then $c(G/e) = c(G) - 1$ and Proposition 2.18 gives $\delta(G/e) \neq \delta(G)$. \square

3. HYPERBOLICITY OF MINOR GRAPHS

In order to obtain results on hyperbolicity of minor graphs we deal now with the other transformation of graphs involved in the definition of minor: the deletion of an edge. Recall that $G \setminus e$ is the graph with $V(G \setminus e) = V(G)$ and $E(G \setminus e) = E(G) \setminus \{e\}$. Theorem 3.3 below provides quantitative relations between $\delta(G \setminus e)$ and $\delta(G)$, where e is any edge of G .

One can expect that the edge deletion is a monotone transformation for the hyperbolicity constant. However, the following examples provide two families of graphs in which the hyperbolicity constant increases and decreases, respectively, by removing some edge.

Example 3.1. Consider G_n as a cycle graph C_n with $n \geq 3$ vertices and fix $e_n \in E(G_n)$. Thus, $G_n \setminus e_n$ is isomorphic to a path graph P_n . Since $\delta(C_n) = n/4$ and $\delta(P_n) = 0$, we have $\delta(G_n) > \delta(G_n \setminus e_n)$ and

$$\lim_{n \rightarrow \infty} (\delta(G_n) - \delta(G_n \setminus e_n)) = \infty.$$

Example 3.2. Let $C_{a,b,c}$ be the graph with three disjoint paths joining two vertices with lengths $a \leq b \leq c$. Consider an edge e of $C_{a,b,c}$ contained in the path with length a . It is easy to check that $\delta(C_{a,b,c} \setminus e) = (c+b)/4$ and [26, Lemma 19] gives that $\delta(C_{a,b,c}) = (c + \min\{b, 3a\})/4$. If $3a < b$, then $\delta(C_{a,b,c}) < \delta(C_{a,b,c} \setminus e)$. In particular, consider $\Gamma_n = C_{n,4n,4n}$ and fix $e_n \in E(\Gamma_n)$ contained in the path with length n . Thus, $\delta(\Gamma_n) = 7n/4$ and $\delta(\Gamma_n \setminus e_n) = 2n$. So, $\delta(\Gamma_n) < \delta(\Gamma_n \setminus e_n)$ and

$$\lim_{n \rightarrow \infty} (\delta(\Gamma_n \setminus e_n) - \delta(\Gamma_n)) = \infty.$$

In [7] the authors obtain quantitative information about the distortion of the hyperbolicity constant of the graph $G \setminus e$ obtained from the graph G by deleting an arbitrary edge e from it. The following theorem is a weak version of their main result.

Theorem 3.3. [7, Theorem 3.15] Let G be a graph and $e = [A, B] \in E(G)$ with $G \setminus e$ connected. Then

$$(3.8) \quad \max \left\{ \frac{1}{5} \delta(G \setminus e), \frac{1}{4} (d_{G \setminus e}(A, B) + 1) \right\} \leq \delta(G) \leq 6\delta(G \setminus e) + d_{G \setminus e}(A, B).$$

Finally, Theorems 2.12 and 3.3 give the following qualitative result.

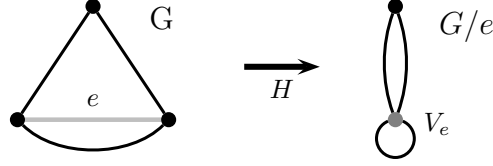
Theorem 3.4. Let G be a graph and G' a (connected) minor graph of G obtained by a finite number of edges removed and/or contracted. Then G is hyperbolic if and only if G' is hyperbolic.

4. NON-SIMPLE GRAPHS

Simple graphs are the usual context in the study of hyperbolicity. However, the operation of contraction is naturally defined for non-simple graphs. For this reason, in this last section we study the distortion of the hyperbolicity constant by contraction of one edge in non-simple graphs.

Since we work with non-simple graphs, if there are $n_1 \geq 1$ edges in G joining v and A and $n_2 \geq 1$ edges joining v and B for some $v \in V(G)$, then we obtain $n_1 + n_2$ edges joining v and V_e in G/e , see Figure 3. Thus, in the context of non-simple graphs a cycle $C \in \mathcal{C}(G, e)$ is transformed in a double edge in G/e , as in Figure 3. Furthermore, for each edge $e' \neq e$ joining A and B in G , we obtain a loop at V_e in G/e .

We define the map $H : G \rightarrow G/e$ in the following way: if x belongs to the edge e , then $H(x) := V_e$; if $x \in G$ does not belong to e , then $H(x)$ is the “natural inclusion map”. Clearly H is onto, i.e., $H(G) = G/e$. Besides, H is an injective map in $G \setminus \{e\}$.

FIGURE 3. The map H .

We prove now a version of Proposition 2.1 for non-simple graphs.

Proposition 4.1. *Let G be a non-simple graph and $e \in E(G)$. Then*

$$(4.9) \quad d_{G/e}(H(x), H(y)) \leq d_G(x, y) \leq d_{G/e}(H(x), H(y)) + 1,$$

for every $x, y \in G$.

Proof. Fix $x, y \in G$. Let $[xy]_G$ be a geodesic in G joining x and y . Clearly, $H([xy]_G)$ is a path joining $H(x)$ and $H(y)$ with length at most $L([xy]_G)$. Hence, we obtain $d_{G/e}(H(x), H(y)) \leq d_G(x, y)$.

Let γ' be a geodesic in G/e joining $H(x)$ and $H(y)$. Then there is a path γ in G with $H(\gamma) = \gamma'$ and $L(\gamma) \leq L(\gamma') + 1$ since γ can contain e or a subset of e . Therefore, $d_G(x, y) \leq L(\gamma) \leq L(\gamma') + 1 = d_{G/e}(H(x), H(y)) + 1$. \square

Note that the inequalities in (4.9) are attained. If G is any non-simple graph, $\{v, w\} \neq \{A, B\}$ and $[v, w] \in E(G)$, then $d_{G/e}(H(v), H(w)) = 1 = d_G(v, w)$. If G is any non-simple graph, then $d_G(A, B) = 1 = d_{G/e}(H(A), H(B)) + 1$.

For any simple path γ' joining two different points in G/e , $H^{-1}(\gamma')$ is either a simple path γ in G or the union of a simple path γ with e . In both cases, we define $H_0^{-1}(\gamma') := \gamma$. If $\alpha' \in \gamma'$, then we define $H_0^{-1}(\alpha') := H^{-1}(\alpha') \cap H_0^{-1}(\gamma')$ (in particular, $H_0^{-1}(V_e)$ can be either A , B or e). Furthermore, if $V_e \notin \gamma'$, then $H_0^{-1}(\gamma') = H^{-1}(\gamma')$ and

$$H|_{H^{-1}(\gamma')}: H^{-1}(\gamma') \longrightarrow \gamma'$$

is a bijective map.

One can check that the following simpler versions for non-simple graphs of Lemmas 2.6, 2.8 and 2.11 hold.

Lemma 4.2. *Let G be a non-simple graph and $e \in E(G)$. Assume that for some $x, y \in G \setminus \{e\}$ there are two geodesics γ_G and $\gamma_{G/e}$ in G and G/e , respectively, joining x, y and $H(x), H(y)$, respectively, such that $L(\gamma_G) = L(\gamma_{G/e}) = L(H(\gamma_G))$ and $e \subset H_0^{-1}(\gamma_{G/e})$. Then we have*

$$d_{G/e}(H(\alpha), \gamma_{G/e}) \leq \delta(G) \quad \forall \alpha \in \gamma_G$$

and

$$d_{G/e}(\alpha', H(\gamma_G)) \leq 2\delta(G) \quad \forall \alpha' \in \gamma_{G/e}.$$

Lemma 4.3. *Let G be a non-simple graph and $e \in E(G)$ such that G/e is not a tree. Let $[xy]_G$ be a geodesic in G joining $x, y \in J(G)$. Assume that $H([xy]_G)$ is not a geodesic in G/e and let $[H(x)H(y)]_{G/e}$ be a geodesic in G/e joining $H(x)$ and $H(y)$. Then we have*

$$d_G(H_0^{-1}(\alpha'), [xy]_G) \leq \delta(G/e) + 1 \leq \frac{7}{3} \delta(G/e), \quad \forall \alpha' \in [H(x)H(y)]_{G/e}$$

and

$$d_G(\alpha, H_0^{-1}([H(x)H(y)]_{G/e})) \leq 2\delta(G/e), \quad \forall \alpha \in [xy]_G.$$

Lemma 4.4. *Let G be a non-simple graph and $e \in E(G)$. Then G is a tree if and only if G/e is a tree.*

Hence, the main results also hold for non-simple graphs.

Theorem 4.5. *Let G be a non-simple graph and $e \in E(G)$. Then*

$$\frac{1}{3} \delta(G/e) \leq \delta(G) \leq \frac{16}{3} \delta(G/e) + 1.$$

Theorem 4.6. *Let G be a non-simple graph and e a cut-edge in G . Then*

$$\delta(G/e) = \delta(G).$$

Theorem 4.7. *Let G be a non-simple graph and G' a (connected) minor graph of G obtained by a finite number of edges removed and/or contracted. Then G is hyperbolic if and only if G' is hyperbolic.*

REFERENCES

- [1] Alonso, J., Brady, T., Cooper, D., Delzant, T., Ferlini, V., Lustig, M., Mihalik, M., Shapiro, M. and Short, H., Notes on word hyperbolic groups, in: E. Ghys, A. Haefliger, A. Verjovsky (Eds.), *Group Theory from a Geometrical Viewpoint*, World Scientific, Singapore, 1992.
- [2] Bandelt, H.-J. and Chepoi, V., 1-Hyperbolic Graphs, *SIAM J. Discrete Math.* **16** (2003), 323-334.
- [3] Bermudo, S., Rodríguez, J. M. and Sigarreta, J. M., Computing the hyperbolicity constant, *Comput. Math. Appl.* **62** (2011), 4592-4595.
- [4] Bermudo, S., Rodríguez, J. M., Sigarreta, J. M. and Vilaire, J.-M., Gromov hyperbolic graphs, *Discr. Math.* **313** (2013), 1575-1585.
- [5] Bermudo, S., Rodríguez, J. M., Sigarreta, J. M. and Tourís, E., Hyperbolicity and complement of graphs, *Appl. Math. Letters* **24** (2011), 1882-1887.
- [6] Carballosa, W., Casablanca, R. M., de la Cruz, A. and Rodríguez, J. M., Gromov hyperbolicity in strong product graphs, *Electr. J. Comb.* **20**(3) (2013), # P2.
- [7] Carballosa, W., Pestana, D., Rodríguez, J. M. and Sigarreta, J. M., Distortion of the hyperbolicity constant of a graph, *Electr. J. Comb.* **19** (2012), # P67.
- [8] Carballosa, W., Portilla, A., Rodríguez, J. M. and Sigarreta, J. M., Planarity and Hyperbolicity in Graphs, to appear in *Graph Comb.* DOI: 10.1007/s00373-014-1459-4
- [9] Carballosa, W., Rodríguez, J. M. and Sigarreta, J. M., New inequalities on the hyperbolicity constant of line graphs, to appear in *Ars Comb.* Preprint in <http://arxiv.org/abs/1410.2941>
- [10] Carballosa, W., Rodríguez, J. M. and Sigarreta, J. M., Hyperbolicity in the corona and join graphs, to appear in *Aequat. Math.* DOI: 10.1007/s00010-014-0324-0
- [11] Carballosa, W., Rodríguez, J. M., Sigarreta, J. M. and Villeta, M., Gromov hyperbolicity of line graphs, *Electr. J. Comb.* **18** (2011), # P210.
- [12] Charney, R., Artin groups of finite type are biautomatic, *Math. Ann.* **292** (1992), 671-683.
- [13] Chen, B., Yau, S.-T. and Yeh, Y.-N., Graph homotopy and Graham homotopy, *Discrete Math.* **241** (2001), 153-170.
- [14] Chepoi, V. and Estellon, B., Packing and covering δ -hyperbolic spaces by balls, *APPROX-RANDOM 2007* pp. 59-73.
- [15] Coudert, D. and Ducoffe, G., Recognition of C_4 -Free and $1/2$ -Hyperbolic Graphs, *SIAM J. Discrete Math.* **28** (2014), 1601-1617.
- [16] Eppstein, D., Squarepants in a tree: sum of subtree clustering and hyperbolic pants decomposition, *SODA'2007, Proceedings of the eighteenth annual ACM-SIAM symposium on Discrete algorithms*, pp. 29-38.
- [17] Gavaille, C. and Ly, O., Distance labeling in hyperbolic graphs, In *ISAAC 2005*, pp. 171-179.
- [18] Ghys, E. and de la Harpe, P., *Sur les Groupes Hyperboliques d'après Mikhael Gromov*. Progress in Mathematics 83, Birkhäuser Boston Inc., Boston, MA, 1990.
- [19] Gromov, M., Hyperbolic groups, in "Essays in group theory". Edited by S. M. Gersten, M. S. R. I. Publ. **8**. Springer, 1987, 75-263.
- [20] Jonckheere, E. and Lohsoonthorn, P., A hyperbolic geometry approach to multipath routing, *Proceedings of the 10th Mediterranean Conference on Control and Automation (MED 2002)*, Lisbon, Portugal, July 2002. FA5-1.
- [21] Jonckheere, E. A., Contrôle du trafic sur les réseaux à géométrie hyperbolique—Vers une théorie géométrique de la sécurité l'acheminement de l'information, *J. Europ. Syst. Autom.* **8** (2002), 45-60.
- [22] Jonckheere, E. A. and Lohsoonthorn, P., Geometry of network security, *Amer. Control Conf. ACC* (2004), 111-151.
- [23] Krauthgamer, R. and Lee, J. R., Algorithms on negatively curved spaces, *FOCS 2006*.
- [24] Krioukov, D., Papadopoulos, F., Kitsak, M., Vahdat, A. and Boguñá, M., Hyperbolic geometry of complex networks, *Physical Review E* **82**, 036106 (2010).
- [25] Kuratowski, K., Sur le problème des courbes gauches en topologie, *Fund. Math.* (in French) **15** (1930), 271-283.
- [26] Michel, J., Rodríguez, J. M., Sigarreta, J. M. and Villeta, M., Hyperbolicity and parameters of graphs, *Ars Comb.* **100** (2011), 43-63.

- [27] Narayan, O. and Saniee, I., Large-scale curvature of networks, *Physical Review E* **84**, 066108 (2011).
- [28] Oshika, K., Discrete groups, AMS Bookstore, 2002.
- [29] Portilla, A., Rodríguez, J. M., Sigarreta, J. M. and Vilaire, J.-M., Gromov hyperbolic tessellation graphs, to appear in *Utilitas Math.* Preprint in <http://gama.uc3m.es/index.php/jomaro.html>
- [30] Portilla, A. and Tourís, E., A characterization of Gromov hyperbolicity of surfaces with variable negative curvature, *Publ. Mat.* **53** (2009), 83-110.
- [31] Rodríguez, J. M., Characterization of Gromov hyperbolic short graphs, *Acta Math. Sinica* **30** (2014), 197-212.
- [32] Rodríguez, J. M., Sigarreta, J. M., Vilaire, J.-M. and Villeta, M., On the hyperbolicity constant in graphs, *Discr. Math.* **311** (2011), 211-219.
- [33] Rodríguez, J. M. and Tourís, E., Gromov hyperbolicity through decomposition of metric spaces, *Acta Math. Hung.* **103** (2004), 53-84.
- [34] Shang, Y., Lack of Gromov-hyperbolicity in small-world networks, *Cent. Eur. J. Math.* **10**(3) (2012), 1152-1158.
- [35] Shang, Y., Non-hyperbolicity of random graphs with given expected degrees, *Stoch. Models* **29**(4) (2013), 451-462.
- [36] Shavitt, Y. and Tanel, T., On internet embedding in hyperbolic spaces for overlay construction and distance estimation, INFOCOM 2004.
- [37] Tourís, E., Graphs and Gromov hyperbolicity of non-constant negatively curved surfaces, *J. Math. Anal. Appl.* **380** (2011), 865-881.
- [38] Wagner, K., Über eine Eigenschaft der ebenen Komplexe, *Math. Ann.* **114** (1937), 570-590.
- [39] Wu, Y. and Zhang, C., Chordality and hyperbolicity of a graph, *Electr. J. Comb.* **18** (2011), # P43.

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